Automated data-driven inverse problem resolution: Applications in microfluidics and epidemiology

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Joint work with Patrice Abry, Nelly Pustelnik, Valérie Vidal, and Samuel Vaiter

January 30, 2026

AI WILD West, Rennes, France

Observation model

$$\boldsymbol{y} \sim \mathcal{B}\left(\boldsymbol{\Phi}\overline{\boldsymbol{x}}\right)$$

- $\boldsymbol{y} \in \mathbb{R}^P$: degraded observations;
- $\overline{\boldsymbol{x}} \in \mathbb{R}^N$: unknown quantity of interest;
- $\boldsymbol{\Phi} : \mathbb{R}^N \to \mathbb{R}^P$: known deformation;
- $\mathcal{B}$: random measurement noise.

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- ill-conditioned, rank deficient $\boldsymbol{\Phi}$

Inpainting

(Guillemot et al., 2013, *IEEE Sig. Process. Mag.*)

Super-resolution

(Marquina et al., 2008, *J. Sci. Comput.*)

Deblurring

(Pan, 2016, *IEEE Trans. Pattern Anal. Mach. Intell.*)

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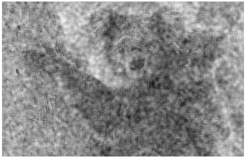
- ill-conditioned, rank deficient $\boldsymbol{\Phi}$
- correlated, data-dependent $\mathcal{B}$

Correlated



(Pascal et al., 2021, *J. Math. Imaging Vis.*)

Data-dependent

(Luisier et al., 2010, *IEEE Trans. Image Process.*)

Multiplicative

(Shama, 2016, *Appl. Math. Comput.*)

Variational estimator

$$\widehat{\boldsymbol{x}}(\boldsymbol{y};\lambda) \in \operatorname*{Argmin}_{\boldsymbol{x}\in\mathbb{R}^N} \mathcal{D}(\boldsymbol{y},\boldsymbol{\Phi}\boldsymbol{x})$$

- $\mathcal{D}(\boldsymbol{y};\cdot) = -\log \mathbb{P}(\boldsymbol{y}|\cdot)$: negative log-likelihood   Ex: $\mathcal{D}(\boldsymbol{y};\boldsymbol{x}) = \|\boldsymbol{y}-\boldsymbol{x}\|_2^2$

No regularization

$\mathcal{R} = 0$

Variational estimator

$$\widehat{\boldsymbol{x}}(\boldsymbol{y};\lambda) \in \underset{\boldsymbol{x}\in\mathbb{R}^N}{\operatorname{Argmin}}\, \mathcal{D}(\boldsymbol{y},\boldsymbol{\Phi x}) + \lambda\mathcal{R}(\boldsymbol{x})$$

- $\mathcal{D}(\boldsymbol{y};\cdot) = -\log\mathbb{P}(\boldsymbol{y}|\cdot)$: negative log-likelihood — Ex: $\mathcal{D}(\boldsymbol{y};\boldsymbol{x}) = \|\boldsymbol{y}-\boldsymbol{x}\|_2^2$
- $\mathcal{R}$: regularization term encoding a priori knowledge — Ex: $\mathcal{R}(\boldsymbol{x}) = \|\mathbf{D}_1\boldsymbol{x}\|_q^q$

(Giovannelli & Idier, 2015, *Wiley*)

No regularization	Smooth	Piecewise constant
		
$\mathcal{R} = 0$	$\mathcal{R}(\boldsymbol{x}) = \|\mathbf{D}_1\boldsymbol{x}\|_2^2$	$\mathcal{R}(\boldsymbol{x}) = \|\mathbf{D}_1\boldsymbol{x}\|_1$
	(Tikhonov et al., 1977, *Wiley*)	(Rudin et al., 1992, *Physica D*)

Fine-tuning of the regularization parameter

Example: $\widehat{\boldsymbol{x}}(\boldsymbol{y};\lambda) \in \operatorname*{Argmin}_{\boldsymbol{x}\in\mathbb{R}^N} \|\boldsymbol{y}-\boldsymbol{x}\|_2^2 + \lambda\|\mathbf{D}_1\boldsymbol{x}\|_2^2$ (Tikhonov)

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not enough

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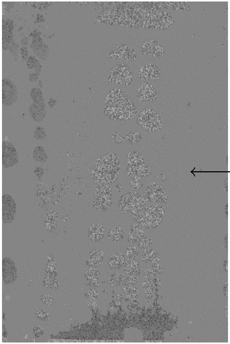
- practical case: ground truth $\overline{\boldsymbol{x}}$ **not available!** $\Longrightarrow$ data-driven $\mathcal{O}(\boldsymbol{y};\lambda)$

Image processing:

Texture segmentation

Multiphase flow through porous media

Laboratoire de Physique, ENS Lyon, V. Vidal, T. Busser, (M. Serres, IFPEN)



Solid
foam

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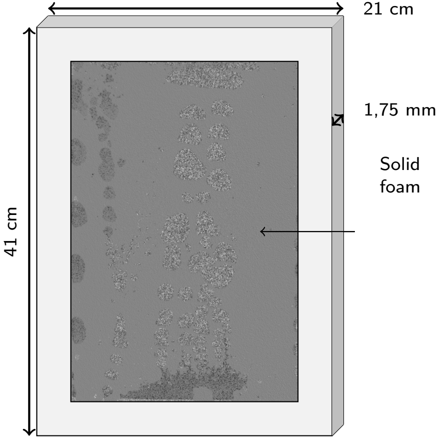

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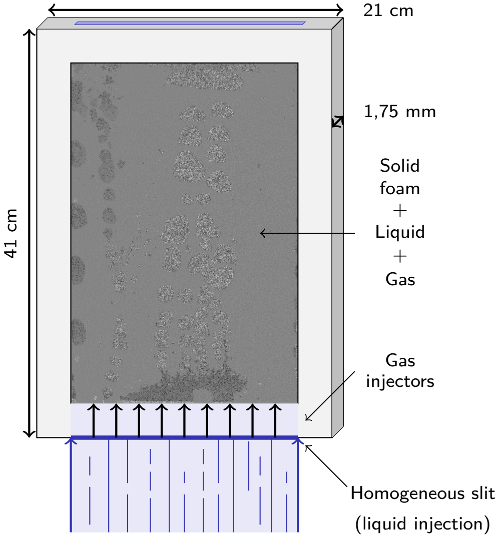

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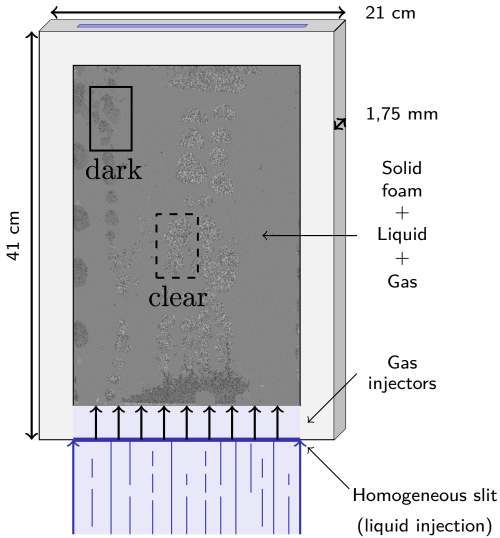

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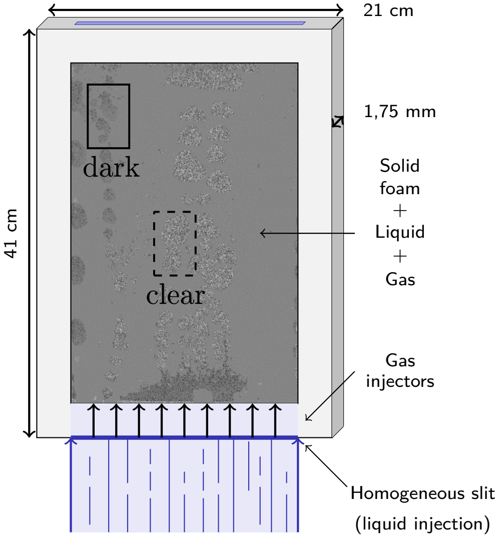
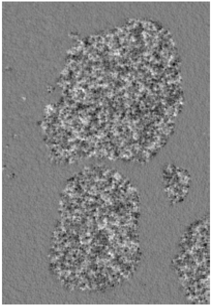
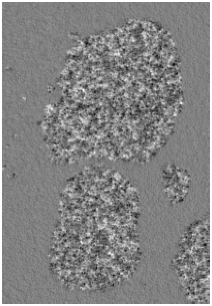
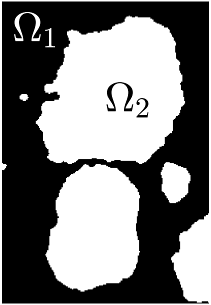


→ 1600 × 1100 pixels
→ video: ∼ 1000 images
→ phase diagram: ∼ 10 flow rates



Textured image segmentation





Goal: obtain a partition of the image into K homogeneous textures

$$\Omega = \Omega_1 \bigsqcup \ldots \bigsqcup \Omega_K$$

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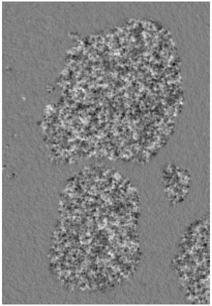

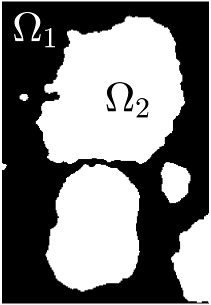

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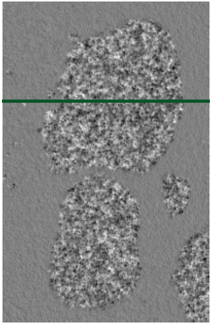
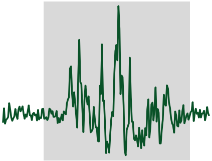
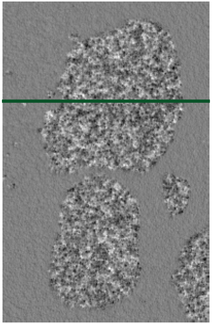
Piecewise monofractal model

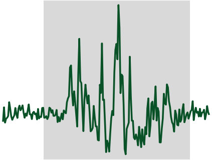




Fractal attributes

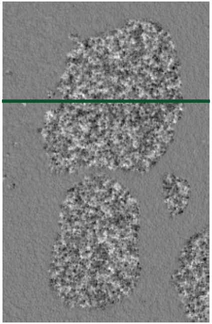
- variance σ^2 *amplitude of variations*

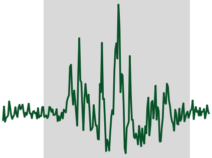




Fractal attributes

- variance σ^2 — *amplitude of variations*
- local regularity h — *scale invariance*





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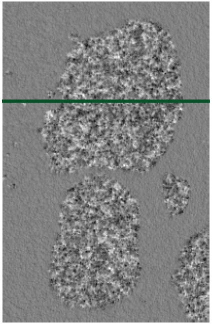
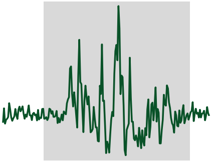
$$|f(x) - f(y)| \leq \sigma(x)|x - y|^{h(x)}$$





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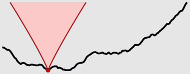
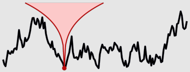
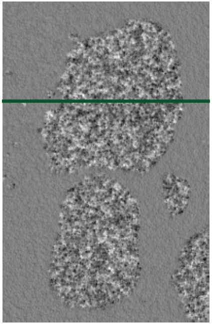
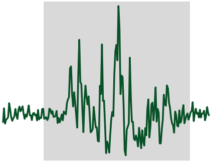


$h(x) \equiv h_1 = 0.9$



$h(x) \equiv h_2 = 0.3$





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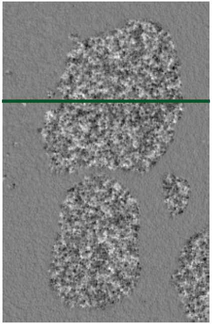
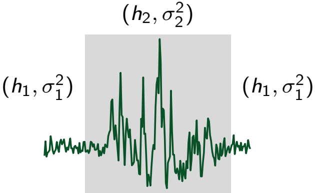
Segmentation

- σ^2 and h piecewise constant





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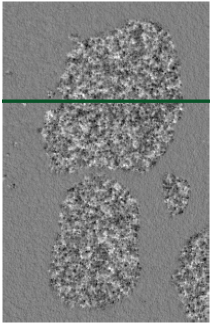
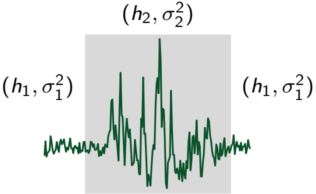
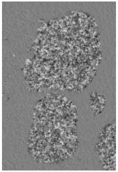




Segmentation

- σ^2 and h piecewise constant
- region Ω_k characterized by (σ_k^2, h_k)

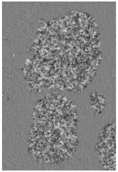
Textured image

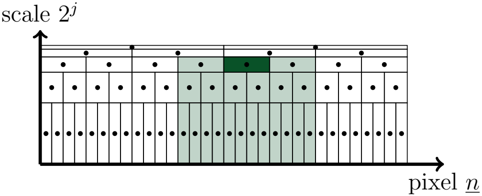


Multiscale analysis

Textured image

Local maximum of wavelet coefficients: $\mathcal{L}_{a,\cdot}$





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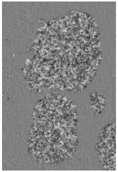

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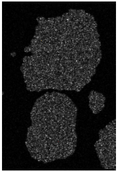

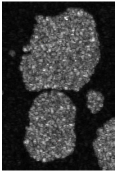

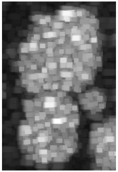

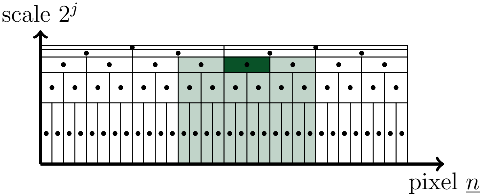

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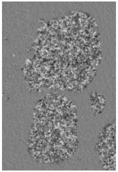

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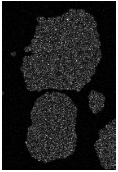
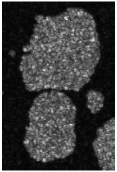
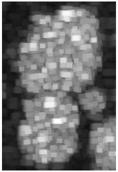




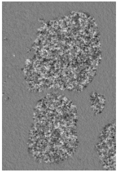


Proposition (Jaffard, 2004, *Proc. Symp. Pure Math.*; Wendt et al., 2009, *Signal Process.*)

$$\log\left(\mathcal{L}_{a,\cdot}\right) \underset{a\to 0}{\simeq} \log(a) \underset{\text{regularity}}{\boldsymbol{h}} + \underset{\substack{\propto \log(\boldsymbol{\sigma}^2) \\ \text{(variance)}}}{\boldsymbol{v}}$$

Multiscale analysis

Textured image



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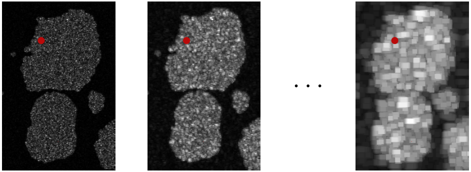

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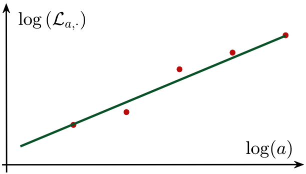

Multiscale analysis

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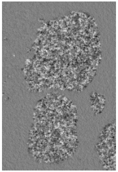

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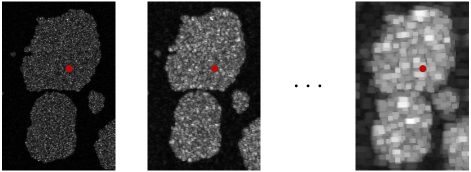
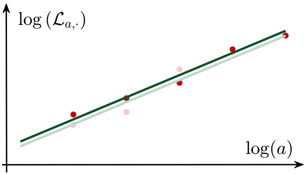


Proposition (Jaffard, 2004, *Proc. Symp. Pure Math.*; Wendt et al., 2009, *Signal Process.*)

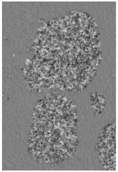
$$\log\left(\mathcal{L}_{a,\cdot}\right) \underset{a \to 0}{\simeq} \log(a) \underbrace{\boldsymbol{h}}_{\text{regularity}} + \underbrace{\boldsymbol{v}}_{\substack{\propto \log(\boldsymbol{\sigma}^2) \\ \text{(variance)}}}$$

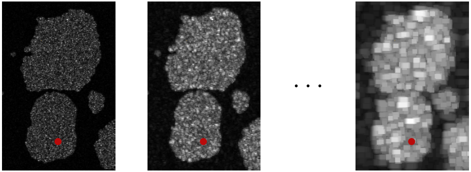


Textured image

Local maximum of wavelet coefficients: $\mathcal{L}_{a,\cdot}$

Scale $a = 2^1$ $a = 2^2$ … $a = 2^5$





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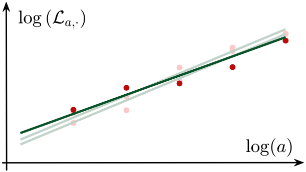
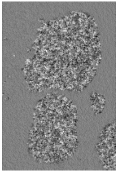
$$\log\left(\mathcal{L}_{a,\cdot}\right) \underset{a\to 0}{\simeq} \log(a) \underbrace{\boldsymbol{h}}_{\text{regularity}} + \underbrace{\boldsymbol{v}}_{\substack{\propto \log(\boldsymbol{\sigma}^2) \\ \text{(variance)}}}$$

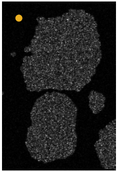


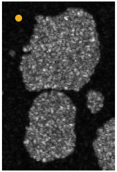
Textured image



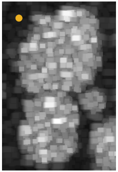
Local maximum of wavelet coefficients: $\mathcal{L}_{a,\cdot}$

Scale $a = 2^1$ $a = 2^2$ … $a = 2^5$





…



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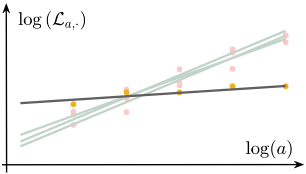
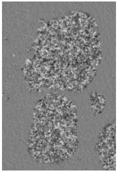


Multiscale analysis

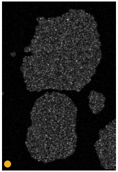
Textured image



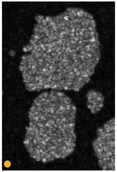
Local maximum of wavelet coefficients: $\mathcal{L}_{a,\cdot}$

Scale

$a = 2^1$

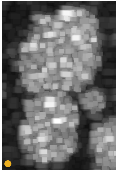


$a = 2^2$



$\cdots$

$a = 2^5$



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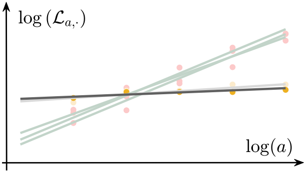
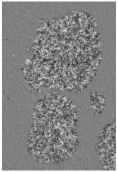
$$\log\left(\mathcal{L}_{a,\cdot}\right) \underset{a\to 0}{\simeq} \log(a) \underbrace{\boldsymbol{h}}_{\text{regularity}} + \underbrace{\boldsymbol{v}}_{\substack{\propto \log(\boldsymbol{\sigma}^2) \\ \text{(variance)}}}$$



Multiscale analysis

Textured image



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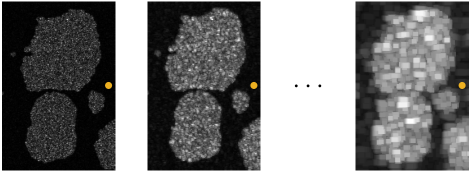
Scale $a = 2^1$ $a = 2^2$ … $a = 2^5$



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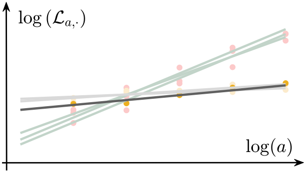
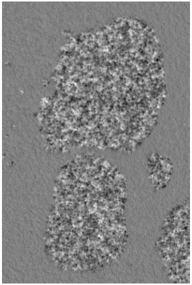


Linear regression $\log\left(\mathcal{L}_{a,\cdot}\right) \simeq \log(a) \underbrace{\boldsymbol{h}}_{\text{regularity}} + \underbrace{\boldsymbol{v}}_{\propto \log(\boldsymbol{\sigma}^2)}$

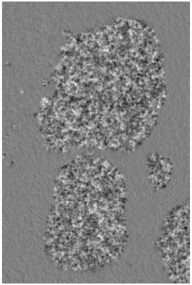
Textured image



Linear regression $\log\left(\mathcal{L}_{a,\cdot}\right) \simeq \log(a) \underbrace{\boldsymbol{h}}_{\text{regularity}} + \underbrace{\boldsymbol{v}}_{\propto \log(\boldsymbol{\sigma}^2)}$

$$\left(\widehat{\boldsymbol{h}}^{\mathrm{LR}}, \widehat{\boldsymbol{v}}^{\mathrm{LR}}\right) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a=a_{\min}}^{a_{\max}} \left\| \log\left(\mathcal{L}_{a,\cdot}\right) - \log(a)\boldsymbol{h} - \boldsymbol{v} \right\|^2$$

Textured image



Linear regression $\log(\mathcal{L}_{a,\cdot}) \simeq \log(a) \underbrace{\boldsymbol{h}}_{\text{regularity}} + \underbrace{\boldsymbol{v}}_{\propto \log(\boldsymbol{\sigma}^2)}$

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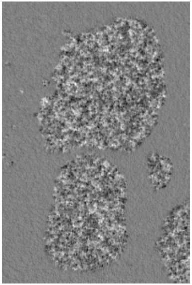
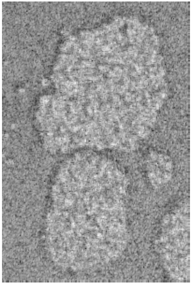
Textured image



Local regularity $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

Local power $\widehat{\boldsymbol{v}}^{\mathrm{LR}}$

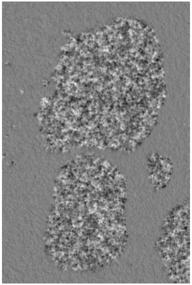


Direct punctual estimation

Linear regression $\underbrace{\mathbb{E}\log\left(\mathcal{L}_{a,\cdot}\right)}_{\text{expected value}} = \log(a)\underbrace{\bar{\boldsymbol{h}}}_{\text{regularity}} + \underbrace{\bar{\boldsymbol{v}}}_{\propto\log(\boldsymbol{\sigma}^2)}$

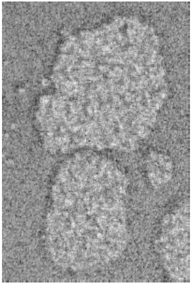
$$\left(\widehat{\boldsymbol{h}}^{\mathrm{LR}}, \widehat{\boldsymbol{v}}^{\mathrm{LR}}\right) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a=a_{\min}}^{a_{\max}} \left\|\log\left(\mathcal{L}_{a,\cdot}\right) - \log(a)\boldsymbol{h} - \boldsymbol{v}\right\|^2$$

Textured image



Local regularity $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

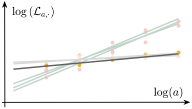
Local power $\widehat{\boldsymbol{v}}^{\mathrm{LR}}$



$\longrightarrow$ large estimation variance

$$\underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}}$$

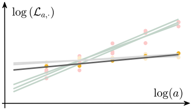
→ fidelity to the log-linear model

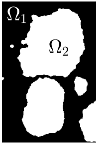


$$\underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}_1\boldsymbol{h}, \mathbf{D}_1\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

→ fidelity to the log-linear model

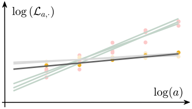
→ favors piecewise constancy



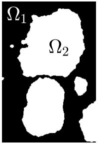


$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \;+\; \lambda \, \underbrace{\mathcal{Q}(\mathbf{D}_1\boldsymbol{h}, \mathbf{D}_1\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

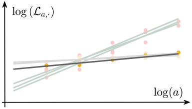
→ fidelity to the log-linear model

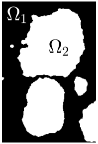


→ favors piecewise constancy



$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\substack{\textbf{Least-Squares} \\ \to \text{ fidelity to the log-linear model}}} + \lambda \underbrace{\mathcal{Q}(\mathbf{D}_1\boldsymbol{h}, \mathbf{D}_1\boldsymbol{v}; \alpha)}_{\substack{\textbf{Total Variation} \\ \to \text{ favors piecewise constancy}}}$$



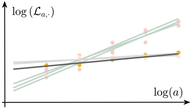


Finite differences $\mathbf{D}_1^{\rightarrow}\boldsymbol{x}$ (horizontal), $\mathbf{D}_1^{\uparrow}\boldsymbol{x}$ (vertical) at each pixel

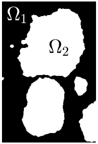
Functionals with either free or co-localized contours

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \underbrace{\sum_a \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}_1\boldsymbol{h}, \mathbf{D}_1\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

→ fidelity to the log-linear model



→ favors piecewise constancy



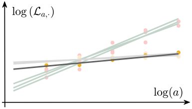
Finite differences $\mathbf{D}_1\boldsymbol{x} = \left[\mathbf{D}_1^{\rightarrow}\boldsymbol{x}, \mathbf{D}_1^{\uparrow}\boldsymbol{x}\right]$

Free: $\boldsymbol{h}$, $\boldsymbol{v}$ are **independently** piecewise constant

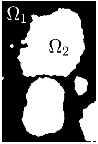
$$\mathcal{Q}_{\mathrm{F}}(\mathbf{D}_1\boldsymbol{h}, \mathbf{D}_1\boldsymbol{v}; \alpha) = \alpha\|\mathbf{D}_1\boldsymbol{h}\|_{2,1} + \|\mathbf{D}_1\boldsymbol{v}\|_{2,1}$$

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \; \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \; + \; \lambda \underbrace{\mathcal{Q}(\mathbf{D}_1\boldsymbol{h}, \mathbf{D}_1\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

→ fidelity to the log-linear model



→ favors piecewise constancy



Finite differences $\mathbf{D}_1\boldsymbol{x} = \left[\mathbf{D}_1^{\rightarrow}\boldsymbol{x}, \mathbf{D}_1^{\uparrow}\boldsymbol{x}\right]$

Free: $\boldsymbol{h}$, $\boldsymbol{v}$ are **independently** piecewise constant

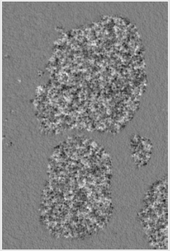
$$\mathcal{Q}_{\mathsf{F}}(\mathbf{D}_1\boldsymbol{h}, \mathbf{D}_1\boldsymbol{v}; \alpha) = \alpha\|\mathbf{D}_1\boldsymbol{h}\|_{2,1} + \|\mathbf{D}_1\boldsymbol{v}\|_{2,1}$$

Co-localized: $\boldsymbol{h}$, $\boldsymbol{v}$ are **concomitantly** piecewise constant

$$\mathcal{Q}_{\mathsf{C}}(\mathbf{D}_1\boldsymbol{h}, \mathbf{D}_1\boldsymbol{v}; \alpha) = \|[\alpha\mathbf{D}_1\boldsymbol{h}, \mathbf{D}_1\boldsymbol{v}]\|_{2,1}$$

$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \quad \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \quad + \quad \lambda \underbrace{\mathcal{Q}(\mathbf{D}_1\boldsymbol{h}, \mathbf{D}_1\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

Textured image

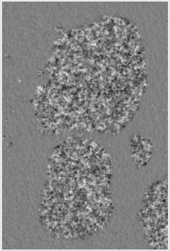


Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

Segmentation *via* iterated thresholding

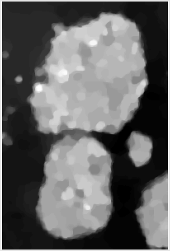
$$\underset{\boldsymbol{h},\boldsymbol{v}}{\text{minimize}} \ \sum_a \underbrace{\|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2}_{\textbf{Least-Squares}} \ + \ \lambda \underbrace{\mathcal{Q}(\mathbf{D}_1\boldsymbol{h}, \mathbf{D}_1\boldsymbol{v}; \alpha)}_{\textbf{Total Variation}}$$

Textured image

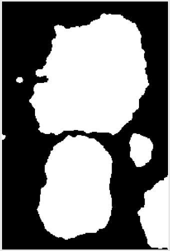


Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

Co-localized contours $\widehat{\boldsymbol{h}}^{\mathrm{C}}$



Threshold estimate† $T\widehat{\boldsymbol{h}}^{\mathrm{C}}$



†(Cai et al., 2013, *J. Sci. Comput.*)

Regularization parameters selection

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h}, \boldsymbol{v}}{\operatorname{argmin}} \sum_a \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}_1 \boldsymbol{h}, \mathbf{D}_1 \boldsymbol{v}; \alpha)$$

Regularization parameters selection

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}_1\boldsymbol{h}, \mathbf{D}_1\boldsymbol{v}; \alpha)$$

Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

$(\lambda, \alpha) = (0, 0)$

Regularization parameters selection

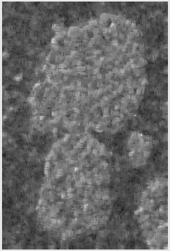
$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,\cdot} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}_1 \boldsymbol{h}, \mathbf{D}_1 \boldsymbol{v}; \alpha)$$

Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

Co-localized contours estimate $\widehat{\boldsymbol{h}}^{\mathrm{C}}$

$(\lambda, \alpha) = (0, 0)$

$(\lambda, \alpha) = (0.5, 0.5)$



too small

Regularization parameters selection

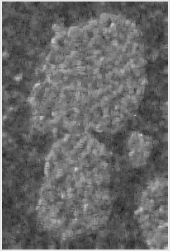
$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}_1 \boldsymbol{h}, \mathbf{D}_1 \boldsymbol{v}; \alpha)$$

Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$

Co-localized contours estimate $\widehat{\boldsymbol{h}}^{\mathrm{C}}$

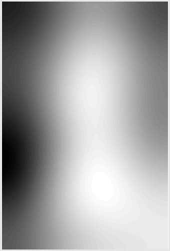
$(\lambda, \alpha) = (0, 0)$

$(\lambda, \alpha) = (0.5, 0.5)$



too small

$(\lambda, \alpha) = (500, 500)$



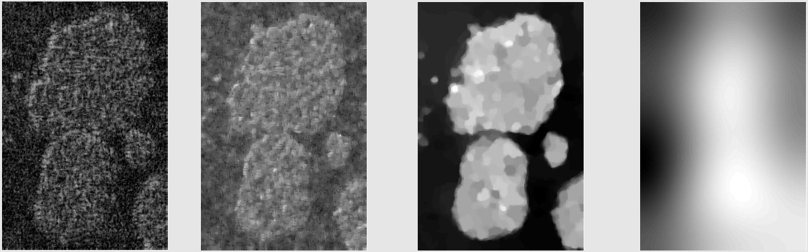
too large

Regularization parameters selection

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\boldsymbol{\mathcal{L}}; \lambda, \alpha) = \underset{\boldsymbol{h},\boldsymbol{v}}{\operatorname{argmin}} \sum_{a} \|\log \boldsymbol{\mathcal{L}}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}_1\boldsymbol{h}, \mathbf{D}_1\boldsymbol{v}; \alpha)$$

Lin. reg. $\widehat{\boldsymbol{h}}^{\mathrm{LR}}$ — Co-localized contours estimate $\widehat{\boldsymbol{h}}^{\mathrm{C}}$

$(\lambda, \alpha) = (0, 0)$ | $(\lambda, \alpha) = (0.5, 0.5)$ | $(\lambda^\dagger, \alpha^\dagger) = (11.5, 0.8)$ | $(\lambda, \alpha) = (500, 500)$



too small | optimal | too large

What *optimal* means? How to determine $\lambda^\dagger$ and $\alpha^\dagger$?

Parameter tuning (Grid search)

$$\left(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{v}}\right)(\mathcal{L}; \lambda, \alpha) = \operatorname*{argmin}_{\boldsymbol{h}, \boldsymbol{v}} \sum_{a} \|\log \mathcal{L}_{a,.} - \log(a)\boldsymbol{h} - \boldsymbol{v}\|^2 + \lambda \mathcal{Q}(\mathbf{D}_1 \boldsymbol{h}, \mathbf{D}_1 \boldsymbol{v}; \alpha)$$

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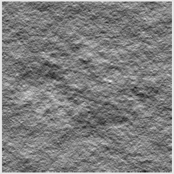
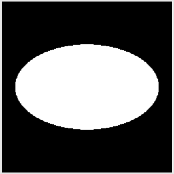
Example





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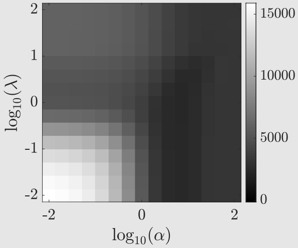

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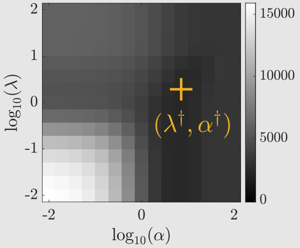

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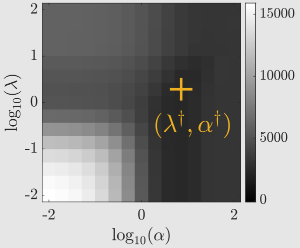

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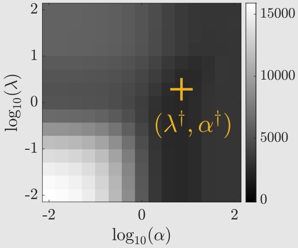

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Stein Unbiased Risk Estimate
(SURE)

Observations $\boldsymbol{y} = \bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}}$: truth and $\zeta \sim \mathcal{N}(\mathbf{0}, \rho^2 \mathbf{I})$

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Parametric estimator $(\boldsymbol{y}; \lambda) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)$

Ex. $$\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) = \begin{cases} \left(\mathbf{I} + \lambda \mathbf{D}^\top \mathbf{D}\right)^{-1} \boldsymbol{y} & \text{(linear)} \\ \underset{\boldsymbol{x}}{\operatorname{argmin}} \, \|\boldsymbol{y} - \boldsymbol{x}\|^2 + \lambda \mathcal{Q}(\mathbf{D}\boldsymbol{x}) & \text{(nonlinear)} \end{cases}$$

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Quadratic error $R(\lambda) \triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \bar{\boldsymbol{x}}\|^2 \overset{?}{=} \mathbb{E}_{\boldsymbol{\zeta}} \widehat{R}(\boldsymbol{y}; \lambda)$ $\bar{\boldsymbol{x}}$ unknown

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Theorem (Stein, 1981,*Ann. Stat.*)

Let $(\boldsymbol{y}; \lambda) \mapsto \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)$ an estimator of $\bar{\boldsymbol{x}}$

- weakly differentiable w.r.t. $\boldsymbol{y}$,
- such that $\boldsymbol{\zeta} \mapsto \langle \widehat{\boldsymbol{x}}(\bar{\boldsymbol{x}} + \boldsymbol{\zeta}; \lambda), \boldsymbol{\zeta} \rangle$ is integrable w.r.t. $\mathcal{N}(\mathbf{0}, \rho^2 \mathbf{I})$.

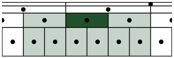
$$\widehat{R}(\boldsymbol{y}; \lambda) \triangleq \|\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) - \boldsymbol{y}\|^2 + 2\rho^2 \operatorname{tr}\left(\partial_{\boldsymbol{y}} \widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda)\right) - \rho^2 P$$
$$\Longrightarrow R(\lambda) = \mathbb{E}_{\boldsymbol{\zeta}}[\widehat{R}(\boldsymbol{y}; \lambda)].$$

Observations $\boldsymbol{y} = \boldsymbol{\Phi}\bar{\boldsymbol{x}} + \boldsymbol{\zeta} \in \mathbb{R}^P$, $\bar{\boldsymbol{x}} \in \mathbb{R}^N$, $\boldsymbol{\Phi} : \mathbb{R}^{P \times N}$ and $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \mathcal{S})$

E.g. the estimators $\widehat{\boldsymbol{h}}(\boldsymbol{\mathcal{L}}; \lambda, \alpha)$ with free or co-localized contours

$\log \boldsymbol{\mathcal{L}} = \boldsymbol{\Phi}(\bar{\boldsymbol{h}}, \bar{\boldsymbol{v}}) + \boldsymbol{\zeta}$ $\quad$ $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \mathcal{S})$ $\quad$ $\mathcal{R} = \|\widehat{\boldsymbol{h}} - \bar{\boldsymbol{h}}\|^2$

$\boldsymbol{\Phi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto \{\log(a)\boldsymbol{h} + \boldsymbol{v}\}_a$



$\boldsymbol{\Pi} : (\boldsymbol{h}, \boldsymbol{v}) \mapsto (\boldsymbol{h}, \mathbf{0})$

Projected estimation error $\quad R_{\boldsymbol{\Pi}}(\boldsymbol{\Lambda}) \triangleq \mathbb{E}_{\boldsymbol{\zeta}} \|\boldsymbol{\Pi}\widehat{\boldsymbol{x}}(\boldsymbol{y}; \boldsymbol{\Lambda}) - \boldsymbol{\Pi}\bar{\boldsymbol{x}}\|^2$

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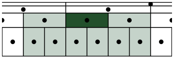

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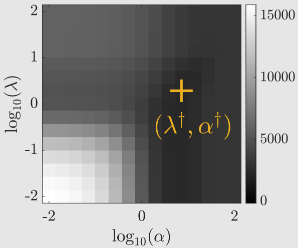

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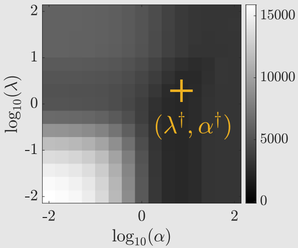

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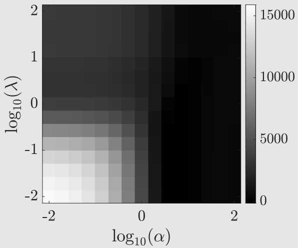

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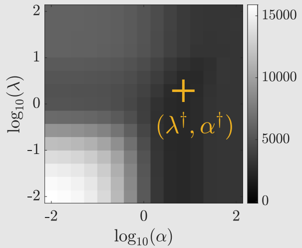

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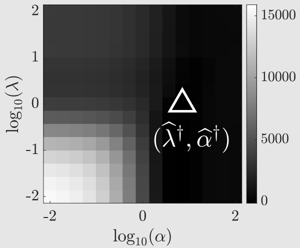

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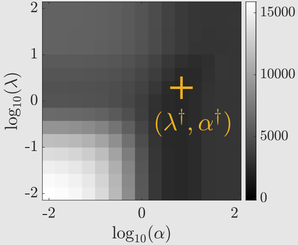

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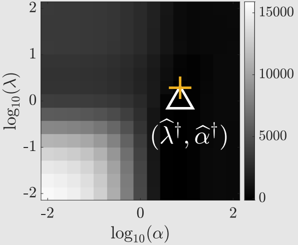

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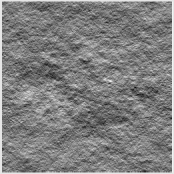
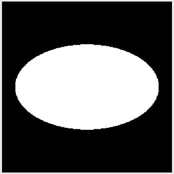
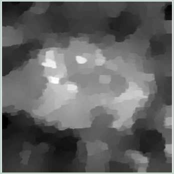




$\widehat{\boldsymbol{h}}^{\mathrm{F}}(\boldsymbol{\mathcal{L}}; \lambda^{\dagger}, \alpha^{\dagger})$
(grid)



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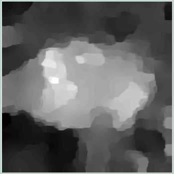
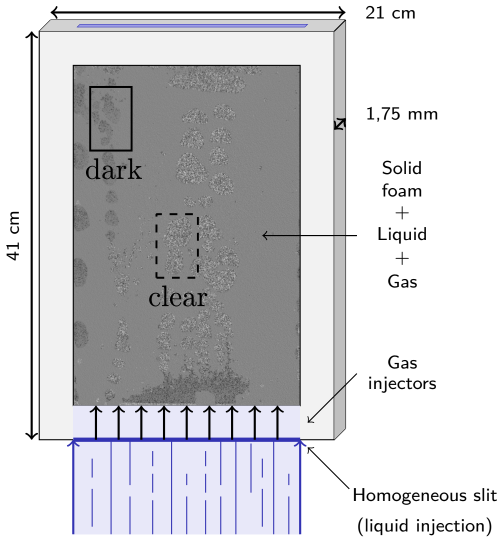


Multiphase flow through porous media

Laboratoire de Physique, ENS Lyon, V. Vidal, T. Busser, (M. Serres, IFPEN)



- → 1600 × 1100 pixels
- → video: ∼ 1000 images
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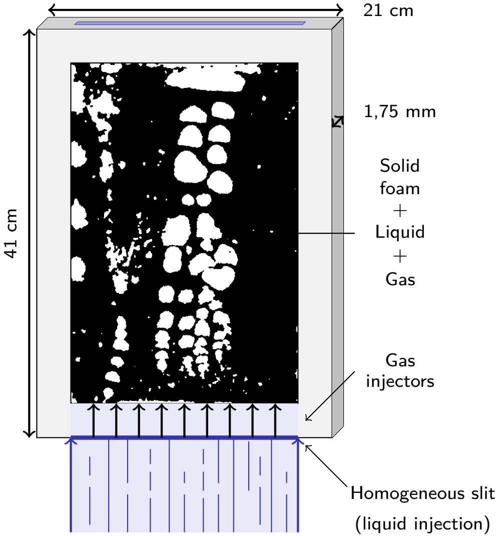

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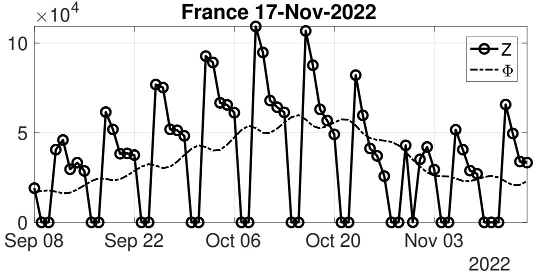
Time series analysis:

Epidemiological indicator estimation

Counts of daily new infections



data from National Health Agencies collected by Johns Hopkins University

$\Longrightarrow$ number of cases not informative enough: need to capture the **dynamics**

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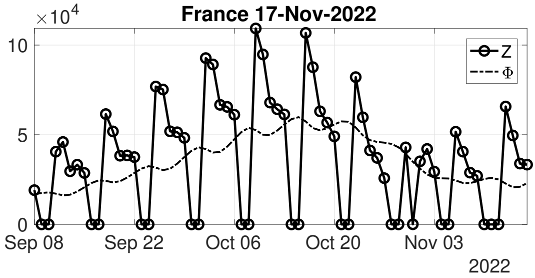

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$\Longrightarrow$ number of cases not informative enough: need to capture the **dynamics**

Design adapted counter measures and evaluate their effectiveness

$\longrightarrow$ efficient monitoring tools *epidemiological model,*

$\longrightarrow$ robust to low quality of the data *managing erroneous counts.*

Reproduction number in Cori model

"averaged number of secondary cases generated by a typical infectious individual"

(Cori et al., 2013, *Am. Journal of Epidemiology*; Liu et al., 2018, *PNAS*)

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$R_t > 1$ the virus propagates at exponential speed,

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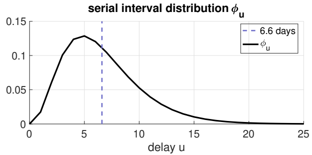
Principle: Z_t new infections at day t

$$\mathbb{E}\left[\mathrm{Z}_t\right] = \mathrm{R}_t \Phi_t, \quad \Phi_t = \sum_{u=1}^{\tau_\Phi} \phi_u \mathrm{Z}_{t-u}$$



with Φ_t global "infectiousness" in the population

$\{\phi_u\}_{u=1}^{\tau_\Phi}$ distribution of delay between onset of symptoms in primary and secondary cases

Gamma distribution truncated at 25 days, of mean 6.6 days and standard deviation 3.5 days

Pandemic study: modeling at the service of monitoring

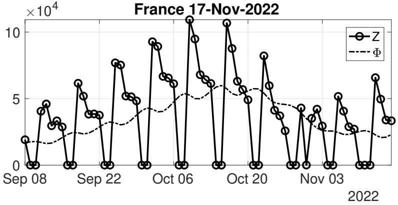
Data: daily counts $\mathbf{Z} = (Z_1, \ldots, Z_T)$

Model: **scaled** Poisson distribution

$$\frac{Z_t}{\gamma} \Big| Z_{t-\tau_\Phi}, \ldots, Z_{t-1}; R_t \sim \mathcal{P}\left(\frac{R_t \Phi_t}{\gamma}\right)$$

$\gamma > 0$ scaling parameter: controls **variance**

(Pascal & Vaiter, 2025, *Signal Process.*)



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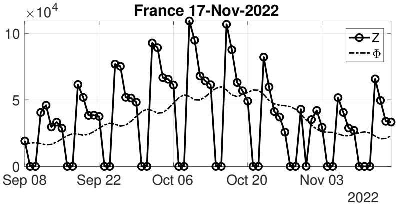

Inverse problem formalism: $\quad \frac{\mathbf{Z}}{\gamma} \sim \mathcal{P}\left(\frac{\mathbf{\Phi R}}{\gamma}\right)$

- $\mathbf{Z} \in \mathbb{N}^T$: reported infection counts,
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- $\mathcal{P}$: data-dependent Poisson noise, with scale parameter γ

$$\Longrightarrow \mathcal{D}(\mathbf{Z}, \mathbf{\Phi R}) = -\log \mathbb{P}(\mathbf{Z}|\mathbf{R})$$

Data: daily counts $\mathbf{Z} = (Z_1, \ldots, Z_T)$

Model: **scaled** Poisson distribution

$$\frac{Z_t}{\gamma} \Big| Z_{t-\tau_\Phi}, \ldots, Z_{t-1}; R_t \sim \mathcal{P}\left(\frac{R_t \Phi_t}{\gamma}\right)$$

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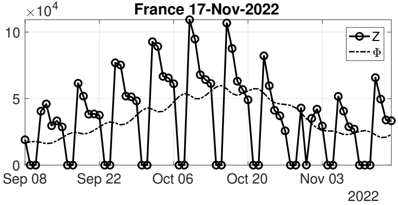

Maximum Likelihood Estimate (MLE)

$$\ln\left(\mathbb{P}(Z_t|\mathbf{Z}_{t-\tau_\Phi:t-1}, R_t)\right)$$

$$\underset{\text{(def.)}}{=} -\frac{1}{\gamma} d_{\mathrm{KL}}(Z_t|R_t\Phi_t) \quad \text{(Kullback-Leibler)}$$

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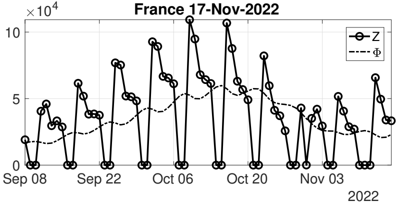

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$$\Longrightarrow \widehat{R}_t^{\text{MLE}} = Z_t / \Phi_t = Z_t / \sum_{u=1}^{\tau_\Phi} \phi_u Z_{t-u}$$

ratio of moving averages

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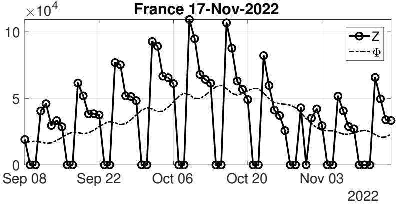

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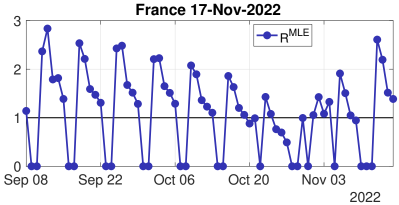

– huge variability along time/ no local trend

– not robust to pseudo-periodicity/ misreported counts

Penalized likelihood: regularization through nonlinear filtering

$$\widehat{\mathbf{R}}^{\mathrm{PKL}} = \underset{\mathbf{R}\in\mathbb{R}_+^T}{\operatorname{argmin}} \sum_{t=1}^{T} \frac{1}{\gamma} \mathrm{d}_{\mathrm{KL}}\left(\mathrm{Z}_t \,|\, \mathrm{R}_t \Phi_t\right) + \lambda \mathcal{R}(\mathbf{R}) \quad \text{(penalized Kullback-Leibler)}$$

with $\mathcal{R}(\mathbf{R})$ favoring some temporal regularity (Abry et al., 2020, *PLOSOne*)

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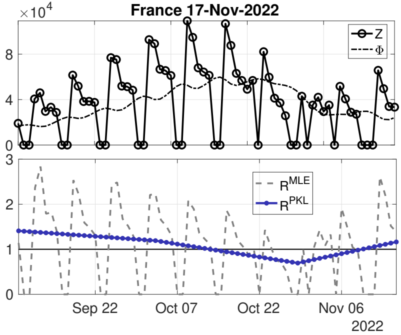
$$\mathcal{R}(\mathbf{R}) = \|\mathbf{D}_2\mathbf{R}\|_1$$

$$(\mathbf{D}_2\mathbf{R})_t = \mathsf{R}_{t+1} - 2\mathsf{R}_t + \mathsf{R}_{t-1}$$

2nd order derivative & ℓ_1-norm

$\Longrightarrow$ piecewise linearity



captures global **trend**, **smooth** temporal behavior, **no pseudo-oscillations**

Penalized Kullback-Leibler estimator:

$$\widehat{\mathbf{R}}(\mathbf{Z};\lambda) = \underset{\mathbf{R}\in\mathbb{R}_+^T}{\operatorname{argmin}} \sum_{t=1}^{T} \frac{1}{\gamma} \mathrm{d}_{\mathrm{KL}}\left(\mathrm{Z}_t \mid \mathrm{R}_t \Phi_t\right) + \lambda \|\mathbf{D}_2 \mathbf{R}\|_1$$

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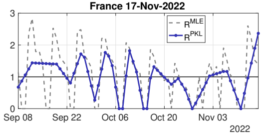
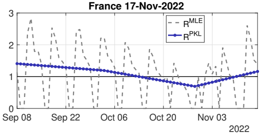
Fine tuning of the regularization parameter:

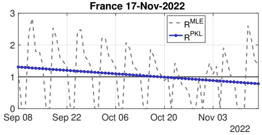
$\lambda = 3.5$



$\boldsymbol{\lambda^\dagger = 50}$



$\lambda = 250$



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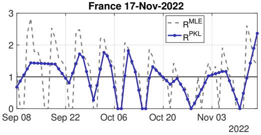

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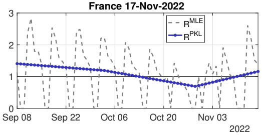

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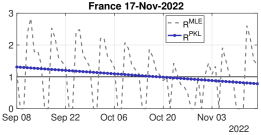

Data-driven oracle minimization

$$\boldsymbol{\lambda}^\dagger \in \underset{\lambda\in\Lambda}{\operatorname{Argmin}}\, \mathcal{O}(\mathbf{Z};\lambda)$$

$\Longrightarrow$ **Goal:** $\mathcal{O}$ data-driven proxy for $\|\widehat{\mathbf{R}}(\mathbf{Z};\lambda) - \overline{\mathbf{R}}\|_2^2$

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$$\frac{\mathbf{Z}}{\gamma} \sim \mathcal{P}\left(\frac{\mathbf{\Phi}\mathbf{R}}{\gamma}\right)$$

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Challenges:

- Poisson model: Stein lemma does not apply (Eldar, 2008, *IEEE Trans. Signal Process.*; Luisier et al., 2010, *IEEE Trans. Image Process.*; Li et al., 2017, *IEEE Trans. Image Process.*)

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  $\Longrightarrow$ Novel counterpart of Stein lemma for driven autoregressive Poisson model

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Autoregressive **P**oisson **U**nbiased **R**isk **E**stimate (APURE)

Data-driven hyperparameter selection on synthetic Poisson data

Model

$$\frac{Z_t}{\gamma} \sim \mathcal{P}\left(\frac{\overline{R}_t \Phi_t(Z_1, \ldots, Z_{t-1})}{\gamma}\right)$$

with scale parameter $\log_{10} \gamma \equiv 3$

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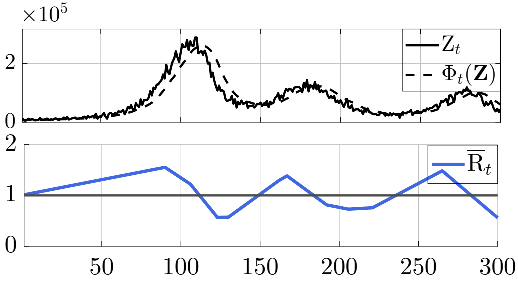

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True prediction $\mathcal{P}$ & estimation $\mathcal{E}$ errors

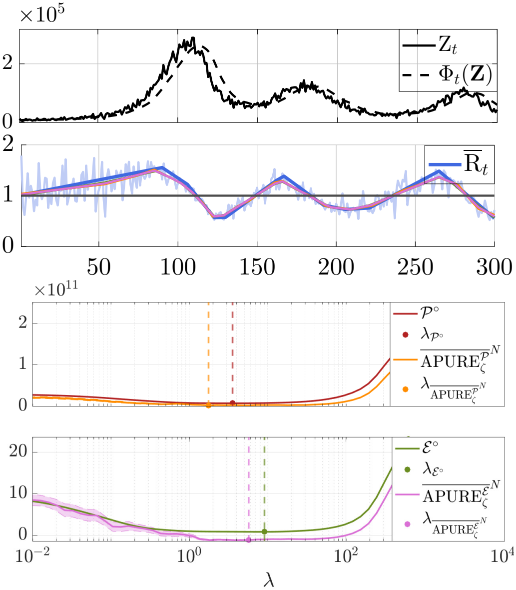
$$\mathcal{P}^\circ(\mathbf{Z};\boldsymbol{\theta}) = \|\widehat{\mathbf{R}}(\mathbf{Z};\boldsymbol{\theta}) \odot \boldsymbol{\Phi}(\mathbf{Z}) - \overline{\mathbf{R}} \odot \boldsymbol{\Phi}(\mathbf{Z})\|_2^2$$

$$\mathcal{E}^\circ(\mathbf{Z};\boldsymbol{\theta}) = \|\widehat{\mathbf{R}}(\mathbf{Z};\boldsymbol{\theta}) - \overline{\mathbf{R}}\|_2^2$$

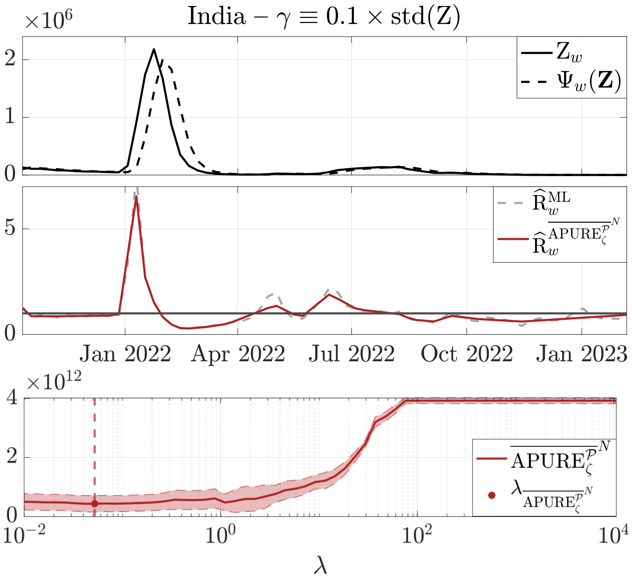
Unbiased risk estimators

$$\overline{\mathrm{APURE}_\zeta^{\mathcal{P}}}^N = \frac{1}{N}\sum_{n=1}^{N} \mathrm{APURE}_{\zeta^{(n)}}^{\mathcal{P}}$$

$$\overline{\mathrm{APURE}_\zeta^{\mathcal{E}}}^N = \frac{1}{N}\sum_{n=1}^{N} \mathrm{APURE}_{\zeta^{(n)}}^{\mathcal{E}}$$



Data-driven hyperparameter selection on weekly COVID-19 counts



Pascal & Vaiter, 2024, *Preprint arXiv:2409.14937*

Codes: github.com/bpascal-fr/APURE-Estim-Epi

Inverse problem $\boldsymbol{y} \sim \mathcal{B}\left(\boldsymbol{\Phi}\overline{\boldsymbol{x}}\right)$

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$\Longrightarrow$ $\mathcal{O}$: Unbiased Risk Estimate (Stein, 1981, *Ann. Stat.*; Eldar, 2008, *IEEE Trans. Signal Process.*; Luisier et al., 2010, *IEEE Trans. Image Process.*; Deledalle et al., 2014, *SIAM J. Imaging Sci.*; Pascal et al., 2021, *J. Math. Imaging Vis.*; Lucas et al., 2023, *Signal, Image Video Process.*)

- Texture segmentation: additive correlated Gaussian noise;
- Epidemic monitoring: driven autoregressive data-dependent Poisson noise.

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Extensions and perspectives

- Efficient and robust scheme for nonconvex $\mathcal{R}(\boldsymbol{x})$;
- Generalization to other noise models: speckle noise in medical imaging;
- Unsupervised learning for $\widehat{\boldsymbol{x}}(\boldsymbol{y}; \lambda) = \mathbf{NN}_{\boldsymbol{\theta}}(\boldsymbol{y})$ with loss $\mathcal{O}(\boldsymbol{y}; \boldsymbol{\theta})$.